

On Timelike Tubular Weingarten Surfaces in Minkowski 3-Space

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Abstract

In this paper, we study the timelike tubular Weingarten surfaces in 3-dimensional Minkowski space IR_1^3 . We have obtained some conditions for being (K_{II}, H) , (K_{II}, K) , timelike tubular Weingarten surfaces where are the second Gaussian curvature the Gaussian curvature and the mean curvature, respectively.

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1 Introduction

Let f and g be smooth functions on a surface M in 3-dimensional Minkowski space IR_1^3 . The existence of a nontrivial functional relation $\Phi(f, g) = 0$ namely the Jacobian determinant of f and g functions $\Phi(f, g) = \det \begin{pmatrix} f_s & f_t \\ g_s & g_t \end{pmatrix} = 0$, where $f_s = \frac{\partial f}{\partial s}$, $f_t = \frac{\partial f}{\partial t}$. A surface is called a Weingarten surface, if there is a nontrivial relation $\Phi(k_1, k_2) = 0$ between its principal curvatures k_1 and k_2 , or, equivalently there is a nontrivial relation $\Phi(K, H) = 0$ between its Gaussian curvature K and mean curvature H . Also we call a surface as a linear Weingarten surface such that the linear combination $aK + bH = c$ is constant along each ruling, where $a, b, c \in \mathbb{R}$, $(a, b, c) \neq (0, 0, 0)$. Several geometers have studied Weingarten and linear Weingarten surfaces. For non-developable surfaces Khnel studied $\{H, K_{II}\}$ and $\{K, K_{II}\}$ Weingarten surfaces [9]. Then, G. Stamou extended this article and gave linear Weingarten surfaces which satisfy $aK_{II} + bH + cH_{II} = d$ is constant along rulings where $a^2 + b^2 + c^2 \neq 0$ [13]. Also in 3-dimensional Minkowski space linear Weingarten surfaces which is foliated by pieces of circles and linear Weingarten helicoidal surfaces under cubic screw motion studied in [4], [5]. F. Dillen and W. Sodsiri examined

$\{K, K_{II}, H, H_{II}\}$ Weingarten and linear Weingarten surfaces in 2005 for the 3-dimensional Minkowski space [1], [2], [3].

A tubular Weingarten and linear Weingarten surface were studied by Ro and Yoon in 3-dimensional Euclidean space E^3 [11]. Karacan and Bukcu constructed tubular surfaces with the assistance of an alternative moving frame [7]. Then geodesics and singular points of tubular surfaces are researched by using one parameter spatial motion along a curve in Minkowski 3-space [6], [8].

In this paper, we give some theorems and conclusions related to timelike tubular Weingarten and linear Weingarten surfaces in 3-dimensional Minkowski space IR_1^3 .

2 Preliminaries

The Minkowski 3-space IR_1^3 is the Euclidean 3-space IR^3 provided with the indefinite inner product given by

$$\langle \cdot, \cdot \rangle = -dy_1^2 + dy_2^2 + dy_3^2,$$

where (y_1, y_2, y_3) is natural coordinates of IR_1^3 . Since $\langle \cdot, \cdot \rangle$ is indefinite inner product, recall that a vector $\beta \in IR_1^3$ can have one of the three causal characters it can be spacelike if $\langle \beta, \beta \rangle > 0$ or $\beta = 0$, timelike if $\langle \beta, \beta \rangle < 0$ and null (lightlike) if $\langle \beta, \beta \rangle = 0$ and $\beta \neq 0$. Similarly, an arbitrary curve $\gamma = \gamma(t)$ in IR_1^3 can locally called as timelike, if its velocity vector $\gamma'(t)$ is timelike. Recall that the norm of a vector is given by $\|\beta\| = \sqrt{|\langle \beta, \beta \rangle|}$ and that the timelike $\gamma(t)$ is said to be of unit speed if $\langle \gamma'(t), \gamma'(t) \rangle = -1$. Moreover, the velocity of curve $\gamma(t)$ is the function $v(s) = \|\gamma'(t)\|$. Denote by $\{t, n, b\}$ the moving Frenet frame along the curve $\gamma(t)$ in the Minkowski space IR_1^3 . Then Frenet formula of $\gamma(t)$ in the space IR_1^3 is defined by [10].

$$\begin{aligned} t' &= \kappa n \\ n' &= \kappa t + \tau b \\ b' &= -\tau n \end{aligned}$$

where the prime denotes the differentiation with respect to t and we denote by κ, τ the curvature and the torsion of the curve γ . Since γ is a timelike curve

$$\begin{aligned} \langle t, t \rangle &= -1, \langle b, b \rangle = \langle n, n \rangle = 1, \\ \langle t, n \rangle &= \langle t, b \rangle = \langle b, n \rangle = 0. \end{aligned}$$

The vector product of the vectors $\beta = (\beta_1, \beta_2, \beta_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ is defined by

$$\beta \wedge \mu = (\beta_3\mu_2 - \beta_2\mu_3, \beta_3\mu_1 - \beta_1\mu_3, \beta_1\mu_2 - \beta_2\mu_1).$$

We denote a timelike surface M in IR_1^3 by

$$x(s, t) = (x_1(s, t), x_2(s, t), x_3(s, t))$$

Let U be the standard unit normal spacelike vector field on a surface M defined by $U = \frac{x_s \wedge x_t}{\|x_s \wedge x_t\|}$, where $x_s = \frac{\partial x(s, t)}{\partial s}$ and $x_t = \frac{\partial x(s, t)}{\partial t}$. Then the first

fundamental form I and the second fundamental form II of a timelike surface M are defined by, respectively

$$I = E ds^2 + 2F ds dt + G dt^2,$$

$$II = e ds^2 + 2f ds dt + g dt^2,$$

where

$$E = \langle x_s, x_s \rangle, \quad F = \langle x_s, x_t \rangle, \quad G = \langle x_t, x_t \rangle$$

$$e = \langle x_{ss}, U \rangle, \quad f = \langle x_{st}, U \rangle, \quad g = \langle x_{tt}, U \rangle.$$

On the other hand, the Gaussian curvature K and the mean curvature H are given by, respectively

$$K = \frac{eg - f^2}{EG - F^2} \langle U, U \rangle$$

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} \langle U, U \rangle.$$

From Brioschi's formula in a Minkowski 3-space, we can compute K_{II} of a surface by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form e, f, g respectively in Brioschi's formula. Consequently, the second Gaussian curvature K_{II} of a non-developable surface is defined by [12]

$$K_{II} = \frac{1}{(eg - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}.$$

3 Timelike Tubular Surfaces of Weingarten Types

Let $\gamma : (a, b) \rightarrow IR_1^3$ be a smooth unit speed timelike curve of finite length which is topologically imbedded in IR_1^3 . The total space N_γ of the normal bundle of $\gamma((a, b))$ in IR_1^3 is naturally diffeomorphic to the direct product $(a, b) \times IR_1^3$ via the translation along γ with respect to the induced normal connection. For sufficiently small $r > 0$, the tubular surface of radius r about the curve γ is the set:

$$T_r(\gamma) = \left\{ \exp_{\gamma(t)} \vartheta \mid \vartheta \in N_{\gamma(t)}, \|\vartheta\| = r, a < t < b \right\}.$$

For a sufficiently small the tubular timelike surface $T_r(\gamma)$ is a smooth surface in IR_1^3 . Then the parametric equation of the timelike tubular surface $T_r(\gamma)$ can be expressed as

$$x(t, \theta) = \gamma(t) + r(\cos \theta n + \sin \theta b) \quad (3.1)$$

Furthermore, we have the natural frame $\{x_t, x_\theta\}$ is given by

$$x_t = (1 + r\kappa \cos \theta)t + r\tau(\cos \theta b - \sin \theta n) = \alpha t + r\tau v, \quad x_\theta = r(\cos \theta b - \sin \theta n) = rv, \quad (3.2)$$

where we put $\alpha = 1 + r\kappa \cos \theta$ and $v = \cos \theta b - \sin \theta n$. From which the components of the first fundamental form are

$$E = -\alpha^2 + r^2\tau^2, \quad F = r^2\tau, \quad G = r^2. \quad (3.3)$$

On the other hand, the unit normal spacelike vector field U is obtained by

$$U = \frac{x_t \wedge x_\theta}{\|x_t \wedge x_\theta\|} = -\cos \theta n - \sin \theta b,$$

from this, the components of the second fundamental form of x are given by

$$e = r\tau^2 - \kappa\alpha \cos \theta, \quad f = r\tau, \quad g = r.$$

If the second fundamental form is non-degenerate, $eg - f^2 \neq 0$, that is, κ, α and $\cos \theta$ are nowhere vanishing. In this case, we can define formally the second Gaussian curvature K_{II} on $T_r(\gamma)$. On the other hand, the Gauss curvature K , the mean curvature H and the second Gaussian curvature K_{II} are given by, respectively

$$K = \frac{\kappa \cos \theta}{r\alpha}, \quad (3.4)$$

$$H = \frac{-(1 + 2r\kappa \cos \theta)}{2r\alpha}, \quad (3.5)$$

$$K_{II} = \frac{1}{4r\alpha^2 \cos^2 \theta} (4r^2 \kappa^2 \cos^4 \theta + 6r\kappa \cos^3 \theta + \cos^2 \theta + 1). \quad (3.6)$$

Differentiating K, H and K_{II} with respect to t and θ , we get

$$K_t = \frac{\kappa' \cos \theta}{r\alpha^2}, \quad K_\theta = -\frac{\kappa \sin \theta}{r\alpha^2}, \quad (3.7)$$

$$H_t = -\frac{\kappa' \cos \theta}{2\alpha^2}, \quad H_\theta = \frac{\kappa \sin \theta}{2\alpha^2}, \quad (3.8)$$

$$\begin{aligned} (K_{II})_t &= \frac{1}{4r\alpha^4 \cos \theta} (2r^3 \kappa^2 \kappa' \cos^4 \theta + 6r^2 \kappa \kappa' \cos^3 \theta + 4r \cos^2 \theta \\ &\quad - 2r^2 \kappa \kappa' \cos \theta - 2r\kappa'), \\ (K_{II})_\theta &= \frac{1}{4r\alpha^4 \cos^4 \theta} (-2r^3 \kappa^3 \cos^6 \theta \sin \theta - 6r^2 \kappa^2 \cos^5 \theta \sin \theta \\ &\quad - 4r\kappa \cos^4 \theta \sin \theta + 4r^2 \kappa^2 \cos^3 \theta \sin \theta \\ &\quad + 6r\kappa \cos^2 \theta \sin \theta + 2 \cos \theta \sin \theta). \end{aligned} \quad (3.9)$$

Now, we investigate a tubular timelike surface $T_r(\gamma)$ in IR_1^3 satisfying the Jacobi equation $\Phi(X, Y) = 0$. By using (3.7) and (3.8), $\Phi(X, Y) = 0$ satisfies identically the Jacobi equation $\Phi(K, H) = K_t H_\theta - K_\theta H_t = 0$. Therefore, $T_r(\gamma)$ is a Weingarten surface. We consider a timelike tubular $T_r(\gamma)$ with non-degenerate second fundamental form in IR_1^3 satisfying the Jacobi equation

$$\Phi(K, K_{II}) = K_t (K_{II})_\theta - K_\theta (K_{II})_t = 0 \quad (3.10)$$

with respect to the Gaussian curvature K and the second Gaussian curvature K_{II} . Then, by (3.7) and (3.9) equation (3.10) becomes

$$r^2 \kappa^2 \kappa' \cos^2 \theta \sin \theta + 2r\kappa \kappa' \cos \theta \sin \theta + \kappa' \sin \theta = 0$$

Since this polynomial is equal to zero for every θ , all its coefficients must be zero. Therefore, we conclude that $\kappa' = 0$. We suppose that a timelike tubular $T_r(\gamma)$

with non-degenerate second fundamental form in IR_1^3 is (H, K_{II}) -Weingarten surface. Then it satisfies the equation

$$H_t (K_{II})_\theta - H_\theta (K_{II})_t = 0, \quad (3.11)$$

which implies

$$r^2 \kappa^2 \kappa' \cos^2 \theta \sin \theta + 2r\kappa\kappa' \cos \theta \sin \theta + \kappa' \sin \theta = 0 \quad (3.12)$$

from (3.12) we can obtain $\kappa' = 0$.

Consequently, we have the following theorems:

Theorem 3.1 *A timelike tubular surface in a Minkowski 3-space is a Weingarten surface.*

Theorem 3.2 *Let $(X, Y) \in \{(K, K_{II}), (H, K_{II})\}$ and let $T_r(\gamma)$ be a timelike tubular surface in Minkowski 3-space with non-degenerate second fundamental form. If $T_r(\gamma)$ is a (X, Y) -Weingarten surface, then the curvature of $T_r(\gamma)$ is a non-zero constant.*

Finally, we study a timelike tubular $T_r(\gamma)$ in IR_1^3 is a linear Weingarten surface, that is, it satisfies the equation

$$aK + bH = c. \quad (3.13)$$

Then, by (3.4) and (3.5) we have

$$(2a\kappa - 2br\kappa - 2r^2c\kappa) \cos \theta - b - 2rc = 0.$$

Since $\cos \theta$ and 1 are linearly independent, we get

$$2a\kappa - 2br\kappa - 2r^2c\kappa = 0, \quad b = -2rc,$$

which imply

$$\kappa(a + cr^2) = 0.$$

If $a + cr^2 \neq 0$, then $\kappa = 0$. Thus, $T_r(\gamma)$ is an open part of a circular cylinder.

Next, suppose that a timelike tubular $T_r(\gamma)$ with non-degenerate second fundamental form in IR_1^3 satisfies the equation

$$aK + bK_{II} = c. \quad (3.14)$$

By (3.4) and (3.6), equation (3.14) becomes

$$(4ar\kappa^2 + 4br^2\kappa^2 - 4cr^3\kappa^2) \cos^4 \theta + (4a\kappa + 6br\kappa - 8cr^2\kappa) \cos^3 \theta + (b - 4cr) \cos^2 \theta + b = 0.$$

Since the identity holds for every θ , all the coefficients must be zero. Therefore, we have

$$\begin{aligned} 4ar\kappa^2 + 4br^2\kappa^2 - 4cr^3\kappa^2 &= 0, \\ 4a\kappa + 6br\kappa - 8cr^2\kappa &= 0, \\ b - 4rc &= 0, \\ b &= 0. \end{aligned}$$

Thus, we get $b = 0, c = 0$ and $\kappa = 0$. In this case, the second fundamental form of $T_r(\gamma)$ is degenerate.

Suppose that a timelike tubular $T_r(\gamma)$ with non-degenerate second fundamental form in IR_1^3 satisfies the equation

$$aH + bK_{II} = c.$$

By (3.5), (3.6) and (3.15), we have

$$(-4ar^2\kappa^2 + 4br^2\kappa^2 - 4cr^3\kappa^2)\cos^4\theta + (-6ar\kappa + 6br\kappa - 8cr^2\kappa)\cos^3\theta + (-2a + b - 4cr)\cos^2\theta + b = 0.$$

from which we can obtain $b = 0$ and $\kappa = 0$.

Consequently, we have the following theorems:

Theorem 3.3 *Let $T_r(\gamma)$ be a timelike tubular surface satisfying the linear equation $aK + bH = c$. If, $a + br \neq 0$, then it is an open part of a circular cylinder.*

Theorem 3.4 *Let $(X, Y) \in \{(K, K_{II}), (H, K_{II})\}$. Then there are no (X, Y) -linear Weingarten tubular in Minkowski 3-space IR_1^3 .*

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